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ANALYSIS OF JET EXHAUSTS WITH REPETITIVE SHOCK STRUCTURE

FINAL REPORT

September 1972

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ANALYSIS OF JET EXHAUSTS WITH REPETITIVE SHOCK STRUCTURE

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Contract DAAH01-71-C-1251

Prepared for Commanding General, U.S. Army Missile Command
Redstone Arsenal, Alabama 35812

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FOREWORD

This document reports the results of an analysis performed by Lockheed's Huntsville Research & Engineering Center under Contract DAAH01-71-C-1251 for the Army Missile Command, Redstone Arsenal, Alabama. The program was monitored by Mr. H. Tracy Jackson of the Missile Systems Laboratory.

SUMMARY

Described herein is an advancement in the state of the art in calculation of the mixing of two concentric (or parallel) streams. The analysis treats the equations of motion governing flow of subsonic or supersonic jets mixing with the surrounding atmosphere.

Several analyses, notably one of AeroChem (Ref. 1) treat the mixing of concentric streams. These analyses, however, assume that there is no lateral pressure gradient. This apriori assumption restricts these analyses to balanced jets which do not vigorously mix or react.

An exhaustive study of full Navier-Stokes solutions and/or simplified mathematical models which retain an elliptic nature did not produce an economically feasible approach to the solution of the jet mixing with shock waves. The approach discussed here was selected as an alternative since inclusion of the radial momentum equation and the treatment of lateral pressure gradients is an improvement over current production calculations which may be achieved with nominal computer run times and storage.

The governing equations of motion are discussed along with the numerical analog and the method of solution. Results of sample calculations are presented and discussed. An appendix attached to the report discusses the computer program and contains an input/output guide.

Although the solution presented is not the final answer to the jet mixing problem it is a necessary step in the evolution of numerical solutions to this complex flow system.

CONTENTS

NOMENCLATURE

Symbol	Description
C_{p_i}	species specific heat at constant pressure
$E_{I, II, III, IV, V}$	Error measure defined in text
f	any function
G	system positive definite error measure
g	local positive definite error measure
H	total enthalpy
h_i	species static enthalpy
p	pressure
q	streamwise velocity
R	universal gas constant
S, n	streamwise and normal coordinates
T	temperature
$W_{I, II, III, IV, V}$	weighting or scaling functions defined in text
y	distance from centerline
Greek	
α	species mass fraction
δ	step modifier
θ	flow angle
μ	eddy viscosity coefficient
ξ	any independent variable
ρ	density of the material
σ	0 for two-dimensional flow; 1 for axisymmetric
ψ	molecular weight
$\dot{\omega}$	species production rate

Section 1
INTRODUCTION

The conventional jet engine expels a stream of hot combustion products into the surrounding atmosphere. These gases radiate in the visible as well as the infrared spectrum. The resultant radiation may be detected by suitable instrumentation and used to track the vehicle.

From a military point of view this has an impact on defensive weapons design (detection and aiming) as well as offensive weapons (protection against detection and aiming by hostile forces). The capability to predict the gas dynamic behavior and salient features of the jet/atmosphere interaction is therefore an important part of the overall problem.

The flowfield structure is very complex and has occupied the efforts of many researchers over the years. Considerable effort has been devoted to each of several important regions within the exhaust such as the near field or base region, the shear layer, the far field fully mixed region and, in the case of supersonic jets, the internal shock structure of the inviscid region of the jet.

Mixed flow (elliptic, hyperbolic) exists in these exhaust structures which tremendously complicates the analysis. Successful, that is practical, analysis must forward march from the jet exit downstream in solving the equations of motion and therefore are of the parabolic or hyperbolic type. The region of interest is so large and the resolution requirements so small that an elliptic solution becomes impractical.

In this study, many methods were investigated in the hopes that the problems could be solved in an economical manner. Some of the numerical techniques investigated were asymptotic time-dependent formulations and relaxation solutions using finite difference and finite element formulations for viscous and

inviscid formulations. No feasible method was found which provided the necessary resolution over the entire region of interest while treating the problem as elliptic. Machine times must be measured in hours, while storage requirements range into hundreds of thousands of words.

A discussion of the state of the art of mixing analysis "Review of Eddy Viscosity Models for Jet Engine Exhaust/Air Mixing" by B. J. Audeh, LMSC-HREC D225588, June 1972 was also produced during the course of this study. The methods discussed are all parabolic because the parabolic analysis offers the best avenue to provide engineering solutions to these problems. Most if not all such analyses assume a balanced jet; i.e., no lateral pressure gradients. It was possible to remove this restriction, however, and thereby strengthen the capability to study jet exhaust/ atmosphere interaction problems. The following discussion is concerned with the technical approach to, and details of, a forward marching solution to the equations of motion with lateral pressure gradients.

Section 2
TECHNICAL DISCUSSION

The analysis of a jet exhausting into the atmosphere is truly a complex procedure. Certain calculations have been produced which under conditions consistent with the basic assumptions have served the designer and analyst well. For supersonic inviscid flow the method of characteristics may be employed to produce a rapid, accurate description of the resultant flow field until a Mach disk is encountered. This type of analysis is generally most useful for highly underexpanded jets in which the Mach disk is many exit radii downstream and far beyond the region of interest.

At the other end of the spectrum, a balanced jet may be reasonably handled by parabolic analyses, a family of solutions to which most mixing programs belong.

Perhaps the most complex condition is the moderately expanded jet in which both the lateral pressure gradients and the mixing at the periphery of the jet are of equal importance. It is technically feasible to employ one or more of the many asymptotic time-dependent solutions or elliptic relaxation procedures to analyze such fields. Several approaches were investigated during the course of this study leading to the conclusion that the technical feasibility is severely compromised, however, by the economics of computer solutions of this type. For the problems of interest to this study, one must be willing to accept several hours of computation time and hundreds of thousands of words of storage on the most advanced machine in order to perform such a calculation.

The approach taken here is to treat the moderately expanded jet as a forward-marching (parabolic) analysis, yet retain the lateral momentum equation. This procedure yields a solution which is rapid, yet retains the important features of the flow field.

2.1 EQUATIONS OF MOTION

The analysis begins by considering the equations of motion normally used in balanced jet mixing analyses but in addition considers the lateral momentum equation. These equations presented in a streamwise normal coordinate system may be found in many texts and in the interest of brevity are merely restated here;

- Global Continuity

$$\rho q \frac{\partial \theta}{\partial n} + \frac{\partial (\rho q)}{\partial S} + \sigma \frac{\rho q \sin\theta}{y} = 0 \tag{1}$$

- Streamwise Momentum

$$\rho q \frac{\partial q}{\partial S} = - \frac{\partial p}{\partial S} + \mu \left\{ \frac{\partial^2 q}{\partial n^2} + \sigma \frac{\cos\theta}{y} \frac{\partial q}{\partial n} \right\} \tag{2}$$

- Normal Momentum

$$\rho q^2 \frac{\partial \theta}{\partial S} + \frac{\partial p}{\partial n} = 0 \tag{3}$$

- Energy

$$\rho q \frac{\partial H}{\partial S} = \mu \left\{ \frac{\partial^2 H}{\partial n^2} + \sigma \frac{\cos\theta}{y} \frac{\partial H}{\partial n} \right\} \tag{4}$$

- Species Continuity

$$\rho q \frac{\partial \alpha_i}{\partial S} = \mu \left\{ \frac{\partial^2 \alpha_i}{\partial n^2} + \sigma \frac{\cos\theta}{y} \frac{\partial \alpha}{\partial n} \right\} + \dot{\omega}_i \tag{5}$$

- State

$$p = \rho R T \sum_{i=1}^{I} \frac{\alpha_i}{\psi} \tag{6}$$

where

$$H = \sum_{i=1}^{I} \alpha_i h_i + q^2/2$$

and the Lewis and Prandtl numbers have been taken as unity.

Thermochemical information exists so that for any species

$$h_i = h_i(T)$$

is known. It will be convenient, however, to refer to the "specific heat" for a species;

$$C_{p_i} = \frac{\partial h_i}{\partial T} \quad .$$

A more convenient form of the energy equation is to express the relations explicitly in terms of temperature. Recalling the definition of enthalpy H, after some rearrangement, the following can be obtained:

$$C_p \frac{\partial T}{\partial S} = \frac{1}{\rho} \cdot \frac{\partial p}{\partial S} + \frac{\mu C_p}{\rho q} \left\{ \frac{\partial^2 T}{\partial n^2} + \sigma \frac{\cos\theta}{y} \frac{\partial T}{\partial n} \right\} + \frac{\mu}{\rho q} \left(\frac{\partial q}{\partial n} \right)^2$$

$$- \frac{1}{\rho q} \sum_{i=1}^{I} h_i \dot{\omega}_i + 2 \frac{\mu}{\rho q} \frac{\partial T}{\partial n} \sum_{i=1}^{I} C_{p_i} \frac{\partial \alpha_i}{\partial n} \tag{7}$$

From the species continuity equation it is evident that species distribution is altered due to the mixing and to species production in the event that chemical reactions are of interest. An excellent finite rate computation produced by AeroChem (Ref. 2) has been used in this analyses to calculate the species production term. The reader is referred to Refs. 1 and 2 for details of the thermochemical calculation. By setting $\dot{\omega}$ to zero a frozen analysis will result.

After the basic partial differential equations which have been chosen to represent the physical situation have been written, a procedure must be developed which solves the equations for the flow parameters. Since a closed form solution is unknown, one must resort to a numerical procedure.

2.2 NUMERICAL ANALOG

A schematic of an idealized coaxial mixing region is shown in Fig. 1.

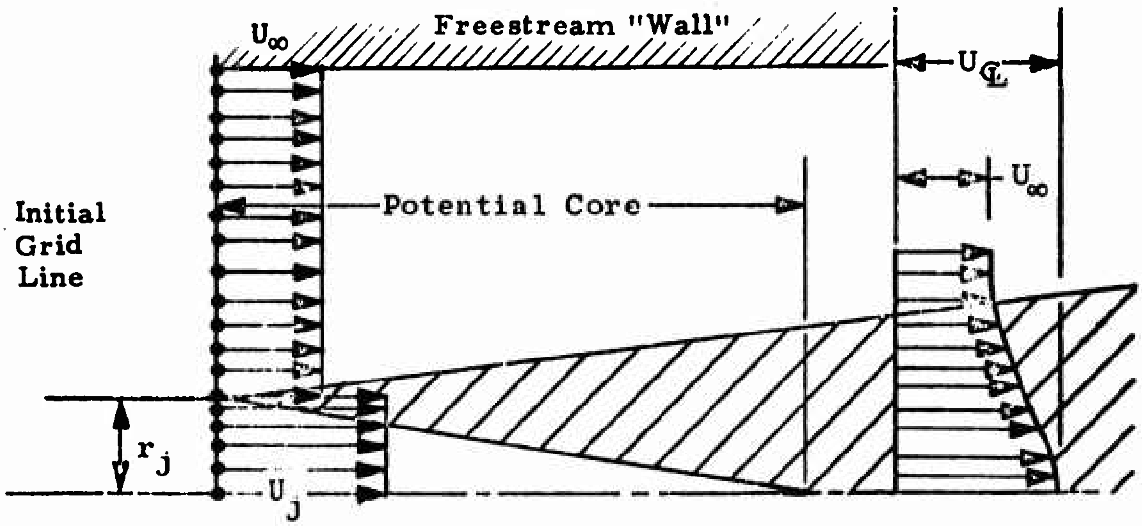


Fig. 1 - Idealized Coaxial Mixing

An initial grid distribution is chosen and a step is taken in the axial direction away from each grid point, creating a new grid line. A typical section of the grid is shown in Fig. 2. Referring to this figure the derivatives of a function

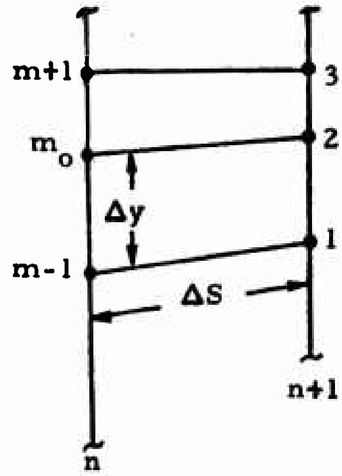


Fig. 2 - Typical Grid Section

will be approximated in the following manner:

$$\frac{\partial f}{\partial S} \simeq \frac{f_{m+1,n+1} - f_{m,n-1}}{\Delta S}; \quad \frac{\partial f}{\partial n} \simeq \frac{f_{m+1,n+1} - f_{m-1,n+1}}{y_{m+1,n+1} - y_{m-1,n+1}}$$

$$\frac{\partial^2 f}{\partial n^2} = \frac{f_{m+1,n+1} + f_{m-1,n+1} - \partial f_{m,n+1}}{1/4\,(y_{m+1,n+1} - y_{m-1,n+1})^2}$$

In addition, all zeroth-order derivatives of f appearing in the governing equations will be approximated by the value of the function at m,n+1.

As previously mentioned, the analysis will be parabolic and this means that, while station n is known in its entirety, station n+1 points must all be solved simultaneously. In order to accomplish this solution of a set of non-linear algebraic equations, the error minimization technique (Ref. 3) will be

employed. This technique creates a positive-definite function of the errors which exist in the flowfield equations and systematically reduces these errors until an acceptable level (convergence) is reached.

Refer to Fig. 2. For simplicity, let the index m, n be represented to zero and the other points as indicated by the appropriate integers. The numerical analogs then become

- Global Continuity

$$\rho_2 q_2 \left(\frac{\partial \theta}{\partial n}\right)_2 + \rho_2 \left(\frac{\partial q}{\partial S}\right)_2 + q_2 \left(\frac{\partial \rho}{\partial S}\right)_2 + \sigma \rho_2 q_2 \frac{\sin\theta_2}{y_2} = E_I \tag{8}$$

- Streamwise Momentum

$$\rho_2 q_2 \left(\frac{\partial q}{\partial S}\right)_2 + \left(\frac{\partial p}{\partial S}\right)_2 - \mu \left\{ \left(\frac{\partial^2 q}{\partial n^2}\right)_2 + \sigma \frac{\cos\theta_2}{y_2} \left(\frac{\partial q}{\partial n}\right)_2 \right\} = E_{II} \tag{9}$$

- Normal Momentum

$$\rho_2 q_2^2 \left(\frac{\partial \theta}{\partial S}\right)_2 + \left(\frac{\partial p}{\partial n}\right)_2 = E_{III} \tag{10}$$

- Energy

$$\rho_2 q_2 C_p \left(\frac{\partial T}{\partial S}\right)_2 - q_2 \left(\frac{\partial p}{\partial S}\right)_2 - \mu C_p \left(\frac{\partial^2 T}{\partial n^2}\right)_2 + \sigma \frac{\cos\theta_2}{y_2} \left(\frac{\partial T}{\partial n}\right)_2 - \mu \left(\frac{\partial q}{\partial n}\right)_2^2$$

$$+ \sum_{i=1}^{I} h_i \dot{\omega}_i - 2\mu \left(\frac{\partial T}{\partial n}\right)_2 \sum_{i=1}^{I} C_{p_i} \left(\frac{\partial \alpha_i}{\partial n}\right)_2 = E_{IV} \tag{11}$$

- Species Continuity

$$\rho_2 q_2 \left(\frac{\partial \alpha_i}{\partial S}\right)_2 - \mu \left\{ \left(\frac{\partial^2 \alpha_i}{\partial n^2}\right)_2 + \sigma \frac{\cos\theta_2}{y_2} \left(\frac{\partial \alpha_i}{\partial n}\right)_2 \right\} - \dot{\omega}_i = E_{Vi} \tag{12}$$

- State

$$p_2 = \rho_2 RT_2 \sum_{i=1}^{I} \frac{\alpha_{i_2}}{\psi_2} \tag{13}$$

2.3 SOLUTION TECHNIQUE

In the error minimization approach, a positive-definite error function is defined at the point 2; vis:

$$g_2 = \frac{1}{2}\left[W_I E_I^2 + W_{II} E_{II}^2 + W_{III} E_{III}^2 + W_{IV} E_{IV}^2 + \sum_{i=1}^{I} W_{Vi} E_{Vi}^2\right] \tag{14}$$

where the W_I, W_{II}, W_{III}, W_{IV}, W_{VI} are scale factors used to nondimensionalize the corresponding equations. The weighting factors were based on the integral values over the entire n station and applied to the (n+1)st station

$$\bar{\rho} = \int_{y=0}^{y_{max}} \rho \, dy/y_{max}, \quad \bar{q} = \int_{y=0}^{y_{max}} q \, dy/y_{max}$$

and

$$W_I = \frac{1}{\bar{\rho}^2 \bar{q}^2}; \quad W_{II} = \frac{1}{\bar{\rho}^2 \bar{q}^4} = W_{III}; \quad W_{IV} = \frac{1}{\bar{\rho}^2 \bar{q}^6}; \quad W_{Vi} = W_I$$

Now the point 2; is in general, any interior point along the n+1 data line so that a positive-definite error function for the entire line can be found by summing all the local gs.

$$G = \sum_{m=2}^{m_t - 1} g_{m,\,n+1} \tag{15}$$

The object is to reduce G to an acceptable level (zero is exact but impractical). This reduction may be made by the method of steepest descent. Consider that the function G is dependent on a set of coordinates ξ_j; then from the calculus

$$dG = \nabla G \bullet d\overline{\xi}$$

where

$$\nabla G = \frac{\partial G}{\partial \xi_1} \overline{e}_{\xi_1} + \frac{\partial G}{\partial \xi_2} \overline{e}_{\xi_2} + \ldots \frac{\partial G}{\partial \xi_J} \overline{e}_{\xi_J}$$

and

$$d\overline{\xi} = d\xi_1 \overline{e}_{\xi_1} + d\xi_2 \overline{e}_{\xi_2} + \ldots + d\xi_J \overline{e}_{\xi_J}$$

Now for a given step length $|d\overline{\xi}|$, the largest change or payoff in G occurs when ∇G and $|d\overline{\xi}|$ are colinear. Then let

$$d\overline{\xi} = \frac{\nabla G}{|\nabla G|} d\xi$$

so that

$$dG = \frac{\nabla G \bullet \nabla G}{|\nabla G|} d\xi$$

or

$$dG = |\nabla G| \, d\xi$$

The desired change in G is from its current level to zero, i.e.,

$$dG \cong \Delta G = 0 - G^k$$

so that

$$|d\xi| = \Delta\xi = \frac{-G^k}{|\nabla G^k|}$$

and

$$\Delta\overline{\xi} = -\frac{G^k \nabla G^k}{|\nabla G^k|^2}$$

The steepest descent recursion formula then becomes

$$\xi^{k+1} = \xi^{k} - \frac{G^{k}\,\nabla G^{k}}{\nabla G \bullet \nabla G^{k}}\,\delta^{k} \tag{17}$$

where a step modifier δ^k has been included to control the descent process. An unconditionally stable solution can be obtained by altering δ^k in an acceptable fashion if the merit or error function increases. That is to say that if

$$G^{k+1} > G^{k},$$

then the step modifier δ^k is halved and the solution tried again ($\overline{\xi}^{k+1}$ is re-evaluated).

To proceed, then the derivatives of G with respect to the independent variables (∇G) must be determined. The parameters or flow properties at station n are fixed so that G must be differentiated with respect to (ρ, q, θ, T, α_i) for all points on the n+1 data line. Take, for instance, the parameter ρ at station m' on the n+1st data line

$$\frac{\partial G}{\partial \rho_{m',n+1}} = \sum_{m=2}^{m_t-1} \frac{\partial g_{m,n+1}}{\partial \rho_{m',n+1}}$$

Because of the three-point influence of the variable ρ_m (the station subscript n+1 is assumed to be understood from this point on) the above expression reduces to

$$\frac{\partial G}{\partial \rho_{m'}} = \frac{\partial g_{m'-1}}{\partial \rho_{m'}} + \frac{\partial g_{m'}}{\partial \rho_{m'}} + \frac{\partial g_{m'+1}}{\partial \rho_{m'}} \tag{18}$$

At this juncture it is expedient to discuss the evaluation of the derivatives of $g_{m'}$ with respect to the surrounding points and reconstruct the above gradient expression later. From Eq. (14) it can be seen that for any ξ

$$\frac{\partial g_m}{\partial \xi} = W_I E_I \frac{\partial E_I}{\partial \xi} + W_{II} E_{II} \frac{\partial E_{II}}{\partial \xi} + W_{III} E_{III} \frac{\partial E_{III}}{\partial \xi}$$

$$+ W_{IV} E_{IV} \frac{\partial E_{IV}}{\partial \xi} + W_{Vi} \sum_{i=1}^{I} E_{Vi} \frac{\partial E_{Vi}}{\partial \xi} \tag{19}$$

The derivatives of E_I with respect to the variable at points 1, 2 and 3 are; (all derivatives not listed are zero)

$$\frac{\partial E_I}{\partial \rho_2} = q_2 \left(\frac{\partial \theta}{\partial n}\right)_2 + \left(\frac{\partial q}{\partial S}\right)_2 + \sigma \frac{q_2 \sin\theta_2}{y_2} + \frac{q_2}{\Delta S}$$

$$\frac{\partial E_I}{\partial q_2} = \rho_2 \left(\frac{\partial \theta}{\partial n}\right)_2 + \frac{\rho_2}{\Delta S} + \sigma \frac{\rho_2 \sin\theta_2}{y_2} + \left(\frac{\partial \rho}{\partial S}\right)_2$$

$$\frac{\partial E_I}{\partial \theta_1} = -\frac{\rho_2 q_2}{2\Delta y} = -\frac{\partial E_I}{\partial \theta_3}; \quad \frac{\partial E_I}{\partial \theta_2} = \sigma \frac{\rho_2 q_2 \cos\theta_2}{y_2}$$

The derivatives of E_{II} are:

$$\frac{\partial E_{II}}{\partial \rho_2} = q_2 \frac{\partial q}{\partial S}_2 ; \quad \frac{\partial E_{II}}{\partial q_1} = -\mu \left\{\frac{1}{\Delta y^2} - \sigma \frac{\cos\theta_2}{2 y_2 \Delta y}\right\}$$

$$\frac{\partial E_{II}}{\partial q_2} = \rho_2 \frac{\partial q}{\partial S}_2 + \frac{\rho_2 q_2}{S} + \frac{2\mu}{\Delta y^2}; \quad \frac{\partial E_{II}}{\partial q_3} = -\mu \left\{\frac{1}{\Delta y^2} + \sigma \frac{\cos\theta_2}{2 y_2 \Delta y}\right\}$$

$$\frac{\partial E_{II}}{\partial \theta_2} = \mu \sigma \frac{\sin\theta_2}{y_2} \left(\frac{\partial q}{\partial n}\right)_2$$

$$\frac{\partial E_{II}}{\partial p_2} = \frac{1}{\Delta S}$$

The derivatives of E_{III} are

$$\frac{\partial E_{III}}{\partial \rho_2} = q_2^2 \left(\frac{\partial \theta}{\partial S}\right)_2 ; \quad \frac{\partial E_{III}}{\partial q_2} = 2\rho_2 q_2 \left(\frac{\partial \theta}{\partial S}\right)_2 ; \quad \frac{\partial E_{III}}{\partial \theta_2} = \frac{\rho_2 q_2^2}{\Delta S}$$

$$\frac{\partial E_{III}}{\partial p_1} = -\frac{1}{2\Delta y} = -\frac{\partial E_{III}}{\partial p_3}$$

The derivatives of E_{IV} are

$$\frac{\partial E_{IV}}{\partial \rho_2} = q_2 C_p \left(\frac{\partial T}{\partial S}\right)_2 ; \quad \frac{\partial E_{IV}}{\partial q_2} = \rho_2 C_p \left(\frac{\partial T}{\partial S}\right)_2 - \left(\frac{\partial p}{\partial S}\right)_2$$

$$\frac{\partial E_{IV}}{\partial q_1} = \frac{\mu}{\Delta y} \left(\frac{\partial q}{\partial n}\right)_2 = -\frac{\partial E_{IV}}{\partial q_3} ; \quad \frac{\partial E_{IV}}{\partial \theta_2} = \frac{\sigma \mu C_p \sin\theta_2}{y_2} \left(\frac{\partial T}{\partial n}\right)_2$$

$$\frac{\partial E_{IV}}{\partial p_2} = -\frac{q_2}{\Delta S} ; \quad \frac{\partial E_{IV}}{\partial T_2} = \frac{\rho_2 q_2 C_p}{\Delta S} + \frac{2 \mu C_p}{\Delta y^2}$$

$$\frac{\partial E_{IV}}{\partial T_1} = -\mu C_p \left\{ \frac{1}{\Delta y^2} - \frac{\sigma \cos\theta}{2 y_2 \Delta y} \right\} + \frac{\mu}{\Delta y} \sum_{i=1}^{I} C_{p_i} \left(\frac{\partial \alpha_i}{\partial n}\right)_2$$

$$\frac{\partial E_{IV}}{\partial T_3} = -\mu C_p \left\{ \frac{1}{\Delta y^2} + \frac{\sigma \cos\theta}{2 y_2 \Delta y} \right\} - \frac{\mu}{\Delta y} \sum C_{p_i} \left(\frac{\partial \alpha_i}{\partial n} \right)_2$$

$$\frac{\partial E_{IV}}{\partial \alpha_{i_2}} = \rho_2 q_2 C_{p_i} \left(\frac{\partial T}{\partial S} \right)_2 - \mu C_{p_i} \left\{ \left(\frac{\partial^2 T}{\partial n^2} \right)_2 + \frac{\sigma \cos\theta_2}{y_2} \left(\frac{\partial T}{\partial n} \right)_2 \right\}$$

$$\frac{\partial E_{IV}}{\partial \alpha_{i_1}} = \frac{\mu}{\Delta y} \left(\frac{\partial T}{\partial u} \right)_2 C_{p_i} = -\frac{\partial E_{IV}}{\partial \alpha_{i_3}}$$

And finally the derivatives of E_{Vi} are

$$\frac{\partial E_{Vi}}{\partial \rho_2} = q_2 \left(\frac{\partial \alpha_i}{\partial S} \right)_2 ; \quad \frac{\partial E_{Vi}}{\partial q_2} = \rho_2 \left(\frac{\partial \alpha_i}{\partial S} \right)_2$$

$$\frac{\partial E_{Vi}}{\partial \alpha_{i_1}} = -\mu \left\{ \frac{1}{\Delta y^2} - \frac{\sigma \cos\theta_2}{2 y_2 \Delta_y} \right\} ; \quad \frac{\partial E_{Vi}}{\partial \alpha_{i_2}} = \frac{\rho_2 q_2}{\Delta S} + \frac{2\mu}{\Delta_y^2}$$

$$\frac{\partial E_{Vi}}{\partial \alpha_{i_3}} = -\mu \left\{ \frac{1}{\Delta y^2} + \frac{\sigma \cos\theta_2}{2 y_2 \Delta y} \right\} ; \quad \frac{\partial E_{Vi}}{\partial \theta_2} = \frac{\mu \sigma \sin\theta_2}{y_2} \left(\frac{\partial \alpha_i}{\partial n} \right)_2$$

where in all of the above $2\Delta y = y_3 - y_1$, while $\Delta y^2 = (y_3 - y_1)^2/4$. Also in the above expressions, the derivatives with respect to p have been taken independently. The equations of state, however, relates p to ρ, T, α_i. The chain rule may be used to account for this relationship. For instance

$$\frac{\partial E_{IV}}{\partial \rho_2} = \left. \frac{\partial E_{IV}}{\partial \rho_2} \right|_{p_2 = \text{constant}} + \frac{\partial E_{IV}}{\partial p_2} \frac{\partial p_2}{\partial \rho_2}$$

where

$$\frac{\partial p_2}{\partial \rho_2} = RT \sum_{i=1}^{I} \frac{\alpha_i}{\psi}$$

In this fashion, the gradient vector can be determined. Components of this vector may exist at the periphery which do not honor or satisfy the boundary conditions.

2.4 BOUNDARY CONDITIONS ENFORCEMENT

The above discussion is concerned with the governing equations of motion controlling the interior region of the flow field. In addition to these relationships, information must be supplied at the boundaries. These conditions select from a large number of possible solutions, that solution which is appropriate to the particular problem. These conditions are not always easy to specify physically or mathematically. Fortunately, in the parabolic mixing analysis the boundary conditions are straightforward.

At large lateral distances, the flow variables must return to the free-stream values. If the data points were initialized to these values then one need only require that

$$\frac{\partial G}{\partial \rho} = \frac{\partial G}{\partial q} = \frac{\partial G}{\partial \theta} = \frac{\partial G}{\partial T} = \frac{\partial G}{\partial \alpha_i} = 0$$

At the inner wall, however, the situation is somewhat more complicated. Although the equations (for the axisymmetric case) have been written with implied symmetry, there is really nothing in the numerical analog to guarantee this. Because of symmetry certain functions are known to be symmetric about the centerline while others are antisymmetric. For instance the axial velocity component, density, temperature, and species are of equal value above and below the centerline while the flow angle is

equal in magnitude but opposite in sign. Therefore, for all but the flow angle (ξ stands for any variable) the boundary condition is

$$\partial\xi/\partial y = 0$$

A second-order Taylor series expansion in this region in which point 1 is on the axis, 2 is just above the axis and 3 is the point above 2. So that

$$\xi_1 = \frac{4}{3}\xi_2 - \frac{1}{3}\xi_3$$

In taking derivatives previously, point 1 was considered independently of points 2 and 3. The symmetry condition, however, causes a dependency. Again using the chain rule this effect may be accounted for by writing

$$\frac{\partial G}{\partial \xi_2} = \left.\frac{\partial G}{\partial \xi_2}\right|_{\xi_1} + \left.\frac{\partial G}{\partial \xi_1}\right|_{\xi_2} \frac{\partial \xi_1}{\partial \xi_2}$$

and

$$\frac{\partial G}{\partial \xi_3} = \left.\frac{\partial G}{\partial \xi_3}\right|_{\xi_1} + \left.\frac{\partial G}{\partial \xi_1}\right|_{\xi_3} \frac{\partial \xi_1}{\partial \xi_3}$$

When the above operation has been completed, the gradient vector has been correctly determined, and the integration using the recursion formula, Eq. (17), can be performed.

Section 3
RESULTS

A typical turbojet engine/atmospheric mixing case is shown as a sample of the program predictions. The case analyzed is a balanced jet case in which the jet velocity is 1800 ft/sec while the freestream velocity is 880 ft/sec. The jet temperature is 900°K and the freestream is 300°K. A constant eddy viscosity model was used with the eddy viscosity chosen to be 0.01. For this example case both streams were assumed to have a molecular weight of 28.02 and frozen (one species) flow was assumed.

Figure 1 shows the velocity distribution versus lateral distance across the mixing zone. Note that the inward mixing rate is much higher than the outer rate. The velocity profiles seem to be affected somewhat by the duct wall or outer boundary condition, and this may affect the inward rate. In a subsequent figure, however, it will be shown that the temperature spread does not act in the same fashion.

The velocity distributions are shown for X = 0, 1, 2, 4, 8 ft from the jet exit plane. Figure 2 illustrates the axial decay of the jet centerline velocity. The initial decay rate appears to be much too fast (eddy viscosity is too large) causing the potential core to disappear almost immediately while the decay rate farther downstream is more realistic indicating that the eddy viscosity chosen is appropriate at large distances.

Figure 3 presents the temperature distribution. The inner and outer spread rates seem to be nearly equal. This is not the same behavior shown by the velocity distribution and would indicate that the duct wall boundary conditions are not affecting the analysis as previously conjectured. There is no valid reason to assume that both spread rates should be the same nor should the temperature mixing rate be the same as that of the velocity.

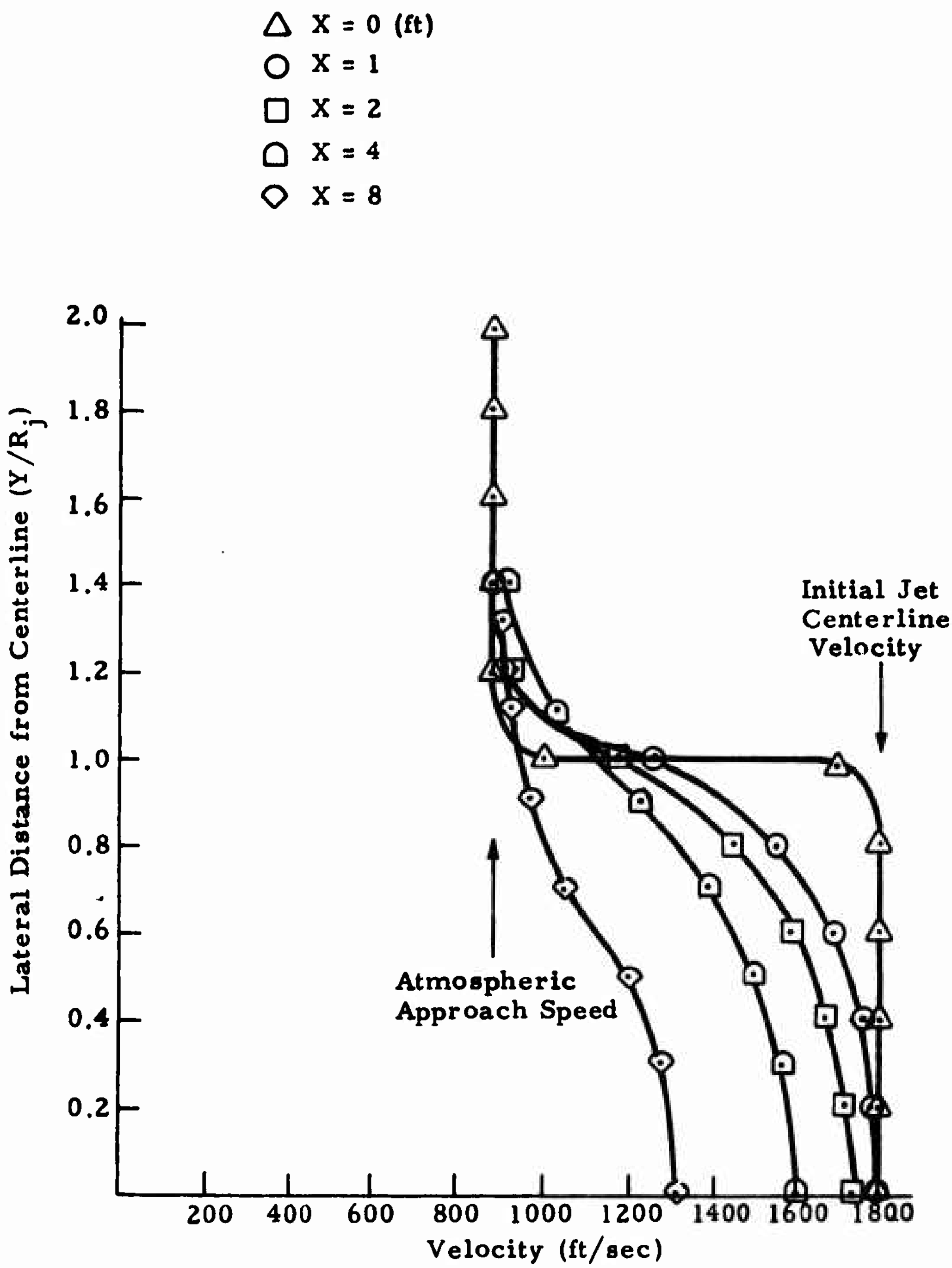


Fig. 1 - Velocity Distribution vs Lateral Distance for Typical Turbojet Mixing Case

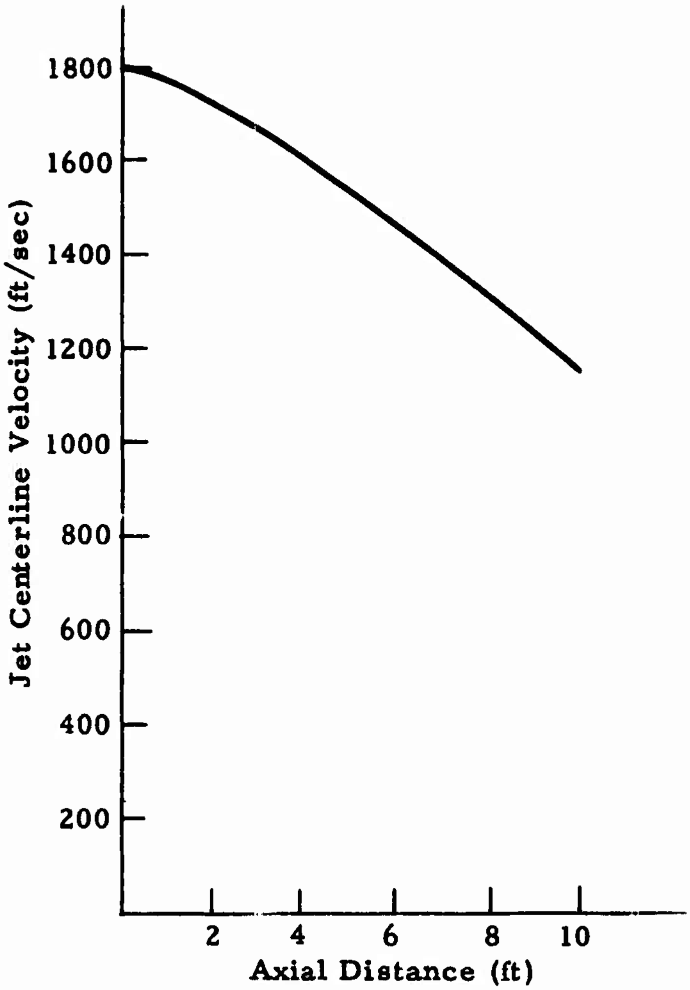


Fig. 2 - Jet Centerline Velocity Decay

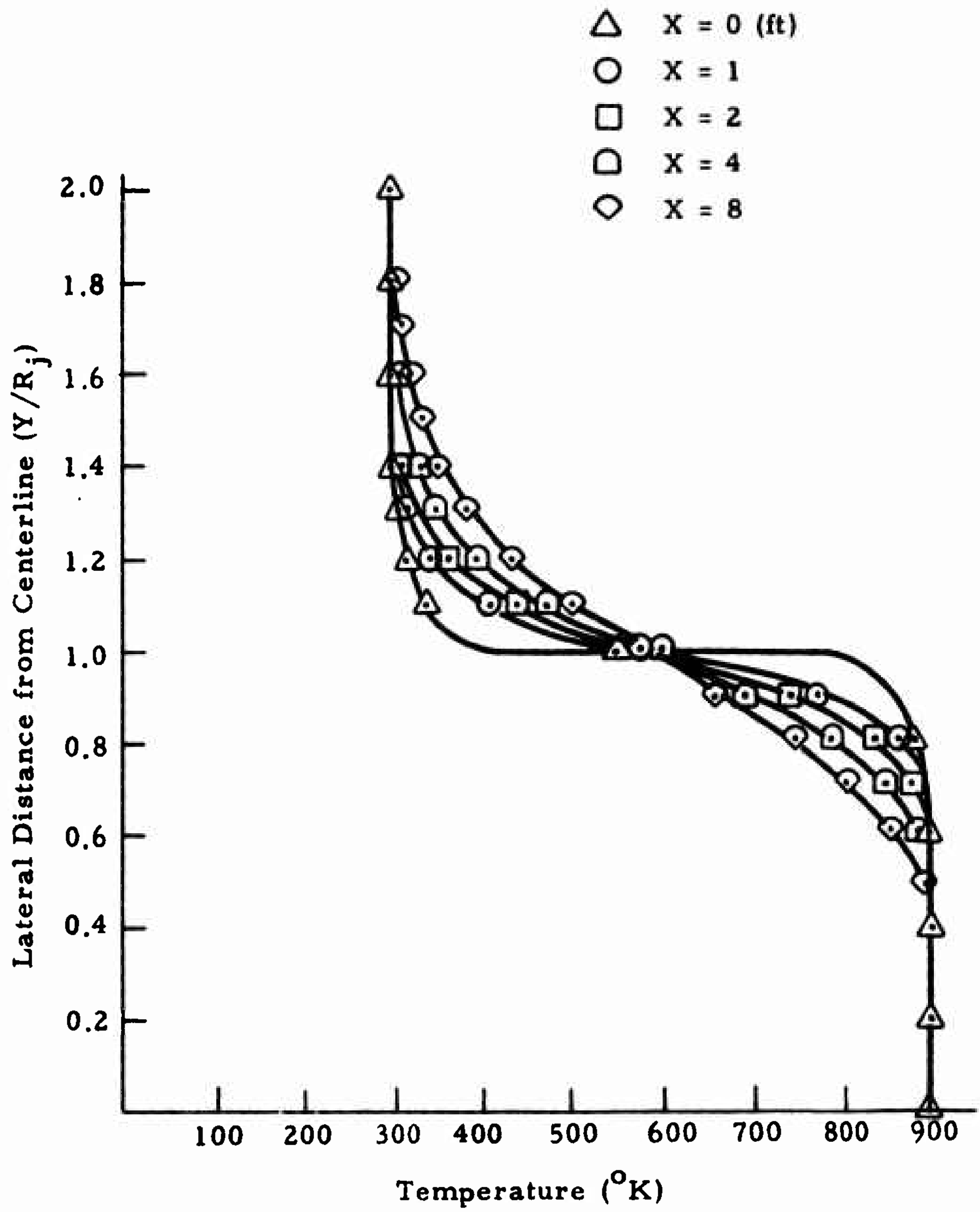


Fig. 3 - Temperature Distribution vs Lateral Distance for Typical Turbojet Mixing

The analysis, being implicit, does not suffer from a stability problem so that large axial steps may be taken and profile definitions may be obtained at large axial distances with nominal computational expense.

Section 4
CONCLUSIONS

The results obtained indicate that the computer program will produce meaningful results in a small amount of computer time. More analysis is required, however, to assess the validity of the calculations for imbalanced jets and reacting mixtures.

In conclusion it is pointed out that repetitive temperature peaks are within the basic capability of the computer program for all speed regimes. The formation of Mach disks, however, being elliptic in nature will not be adequately predicted although it is possible that one of the several approximate Mach disk location theories may be used to form the disk and that the calculation could then be resumed in the parabolic fashion.

Section 5
REFERENCES

1. Mikatarian R. R., and H. S. Pergament, "AeroChem Axisymmetric Mixing with Nonequilibrium Chemistry Computer Program," TP-200, AeroChem Research Laboratories, Inc., Princeton, N. J., June 1969.

2. Komianos, S. A., G. D. Bleich and H. S. Pergament, "The AeroChem Nonequilibrium Streamline Program," TN-103, AeroChem Research Laboratories, Inc., Princeton, N. J., May 1967.

3. Prozan, R. J., and D. E. Kooker, "The Error Minimization Technique with Application to a Transonic Nozzle Solution," J. Fluid Mech., Vol. 43, 1970.

Appendix
COAXIAL JET MIXING WITH LATERAL PRESSURE GRADIENTS COMPUTER PROGRAM

Appendix

1.0 INTRODUCTION

The computer program for "Coaxial Jet Mixing with Lateral Pressure Gradients" is written in FORTRAN V for the CDC 6600 computer at the Army Missile Command Computation Center at Redstone Arsenal. Little or no difficulty is anticipated in program utilization at other facilities with different machines. This appendix is concerned with the computer-oriented aspects of the analysis as opposed to the theoretical considerations which are discussed in the main body of the report.

2.0 DISCUSSION

2.1 Subroutine Description

The program is arranged in logical subdivisions or subroutines. The routines, other than system routines, are MAIN, OUTPUT, LOGIC, EMT, and INOUT. The primary function of each routine is given below:

MAIN – In this routine the basic data and run control information are read and the initial data surface for the forward marching solution is prepared. The routine also computes thermodynamic properties and species production rate. An initial guess for the solution at the next (downstream) surface is prepared and the flow solution subroutines are called. Upon termination of the flow solution for the new surface, control is returned to MAIN.

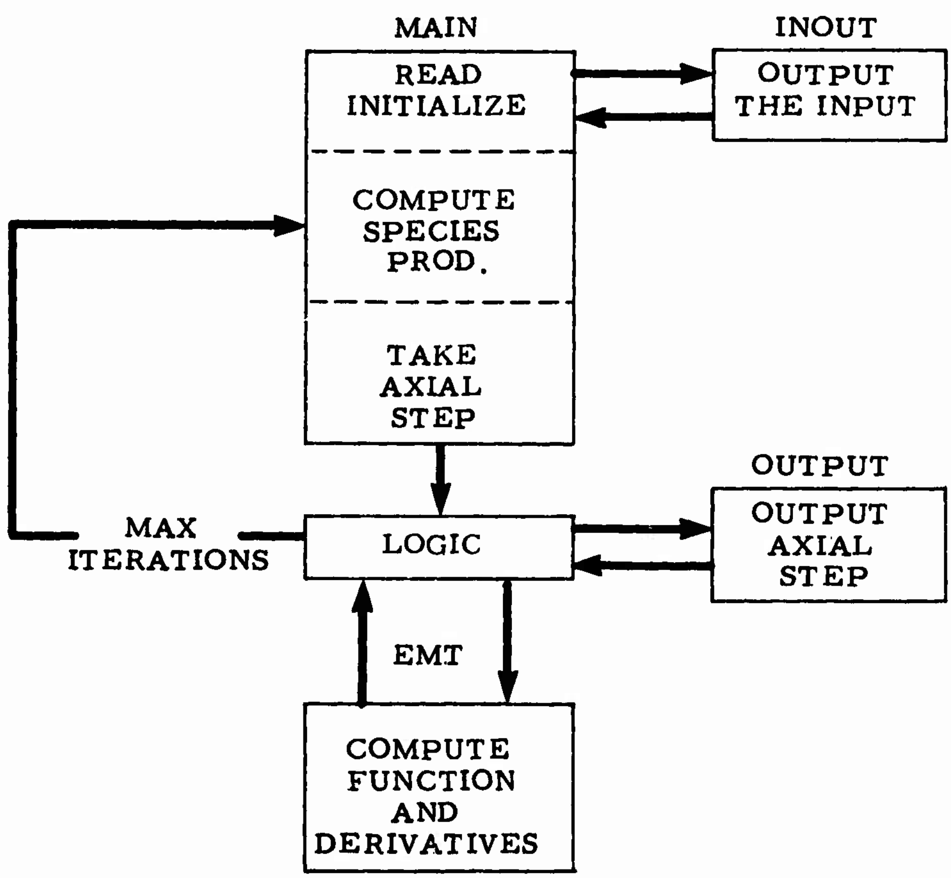
INOUT – This subroutine outputs the input values

LOGIC – This subroutine controls the descent process. Based on the gradient values determined in EMT a step length of each variable in the field

is determined and the next approximation to the flow variables is made. In the event that the merit function increases rather than decreases the step length is halved. A cyclic process is used rather than steepest descent in that only velocity changes are considered until a successful step is made, then temperature, flow angle, density, and species are processed. After a cycle has been completed it is repeated until the maximum number of iterations have been performed. Control is returned to MAIN and the next marching step is taken.

EMT — The merit function and its derivatives are calculated in this subroutine. See the main body of the report for a discussion of these calculations.

OUTPUT — This routine outputs the current axial step flow properties. The overall program flow is given below:



2.2 INPUT INSTRUCTIONS

CARD 1	FORMAT	(12A6)	
	FIELDS 1-12	Any Hollerith information, heading printed atop each page	
CARD 2	FORMAT	(16I5) - Run control parameters	
	FIELD 1	Number of radial stations in jet stream	
	FIELD 2	Number of radial stations in atmospheric stream	
	FIELD 3	Number of input cards with jet properties equal 1 for uniform jet setup equal FIELD1 if distribution across jet desired	
	FIELD 4	Number of input cards with atmospheric properties equal 1 for uniform atmospheric stream equal FIELD2 if distribution desired	
	FIELD 5	Integer selecting type of eddy viscosity model 1 - laminar (Sutherland's) model 2 - constant viscosity model 3 - Ferri viscosity model 4 - Ting-Libby viscosity model 5 - Ting-Libby after mixing region intersects axis 6 - GASL mixing width model	
	FIELD 6	Number of species present across entire region	
	FIELD 7	0 two-dimensional, 1 for axisymmetric analysis	
	FIELD 8	Number of relaxation steps/print step	
	FIELD 9	Maximum number of iterations	
	FIELD 10	Number of reactions (zero if frozen calculation)	
	FIELD 11-16	Not presently used	
CARD 3	FORMAT	7E10.6	
	FIELD 1	Axial coordinate of initial surface	
	FIELD 2	Axial step length	
	FIELD 3	Maximum axial distance	
	FIELD 4	Initial jet radius	

CARD 3	FORMAT 7E10.6	
	FIELD 5	Initial outer edge of atmospheric stream
	FIELD 6	Normalizing dimension (usually jet radius)
	FIELD 7	Eddy viscosity or coefficient as appropriate
CARD GROUP(S) 4	FORMAT (7E10.6) - As many card groups as in FIELD3 of CARD 2 - the first card group pertains to jet centerline	
CARD 1	FIELD 1	Pressure (atm) of jet
	FIELD 2	Temperature (^{o}K) of jet
	FIELD 3	Velocity (ft/sec) of jet
	FIELD 4	Angle (deg) of jet
	FIELD 5	Radial coordinate associated with these properties
CARD(S) 2	FIELDS(1-NS)	Species mole fractions at this location
CARD GROUP(S) 5	FORMAT (7E10.6) - As many cards as in FIELD4 of CARD 2 - the first card pertains to the freestream edge just above the jet	
CARD 1	FIELD 1	Pressure (atm) of atmosphere
	FIELD 2	Temperature (^{o}R) of atmosphere
	FIELD 3	Velocity (ft/sec) of atmosphere
	FIELD 4	Angle (deg) of atmosphere
	FIELD 5	Radial coordinate associated with these properties
CARDS(2)	FIELD(1-NS)	Species mole fractions at this location
CARD GROUP(S)6	FORMAT (A6/4E16.8/4E16.8/4E16.8) - Species information. 1 card group for each species	
	FIELD 1	Species name
	FIELD 2	Molecule weight
	FIELD 3	C_g
	FIELD 4	C_S
	FIELD 5	C_H
	FIELD 6-13	$C_1, \ldots, C_8$

where

C_S = reference entropy, C_H = reference enthalpy

$$C_p = \sum_{i=-4}^{i=4} \bar{C}_i T^i$$

and

$$C_1 = \bar{C}_{-4}; \quad C_2 = \bar{C}_{-3}; \text{ etc.}$$

CARD(S) 7	FORMAT		(A6, 1X, A6, 8X, A6, 1X, A6, 1X, A6, 7X, I2, I1, E8.2, F4.1, F9.1) read in NR such cards in any order	
	FIELD	1	Name of species A	A6
		2	Name of species B	A6
		3	Name of species C	A6
		4	Name of species D (or M)	A6
		5	Name of species E (or M)	A6
		6	Reaction type 1-10 (Ref. 1)	I2
		7	Rate constant type (Ref. 1)	I1
		8	Pre-exponentional factor (cm-particle-sec)	E8.2
		9	Temperature exponent	F4.1
		10	Activation energy cal/mole	F9.1

The above input defines the reaction and its rate. The reaction equation is

$$A + B \rightarrow C + D + E$$

while the rate equation is $k = a\,T^{-b}\, e^{-c/RT}$.

SAMPLE INPUT

```
                      TYPICAL  TURBOJET  AIRCRAFT  -  SEA LEVEL  CONDITIONS
   11    10     1     1     2     1     1    50    50     0
+0       +00+1       +00+3       +02+1       +01+2       +01+1       +01+1       -02
+1       +01-9       +03+18      +04+0       +00+0       +00
+1       +01
+1       +01+3       +03+1       +01+0       +00+3       +01
+1       +01
N2
 0.28015999E+02-0.29290771 E-13  0.19057564E+03 -0.16357003E+06
 -0.57874209E+12 0.51596354 E+10 -0.17279778E+08  0.28151013E+05
 -0.18451609E+02 0.14280968 E-01 -0.39420453E-05  0.54408708E-09
```

SAMPLE OUTPUT

Page 1

```
              LOCKHEED HUNTSVILLE RESEARCH + ENGINEERING CENTER
              AXIS- OR 2D MIXING WITH NON-EQUILIBRIUM CHEMISTRY
              AND VIBRATIONAL ENERGY EXCHANGES      ,INCLUDING
              AXIAL AND LATERAL PRESSURE GRADIENTS

        TYPICAL TURBOJET AIRCRAFT - SEA LEVEL CONDITIONS -  EMT MIXING SMOOTH

  NOZZLE RADIUS= 1.0000000E+00 FEET

  X INITIAL(FEET)= 0.                         X FINAL(FEET)= 1.0000000E+02

  PRINT INCREMENT=  1.0000000E-01             INITIAL STEP SIZE= 1.0000000E-01

  CONSTANT VISCOSITY MODEL  MU= 1.0000000E-03

                                    JET                EDGE
  TEMPERATURE(DEG. KELVIN)      9.0000000E+02      3.0000000E+02
  VELOCITY    (FEET/SECOND)     1.8000000E+03      1.0000000E+00
  PRESSURE    (ATMOSPHERES)     1.0000000E+00      1.0000000E+00
  FLOW ANGLE     (DEGREES)      0.                 0.
  MOLE FRACTION    N2           1.0000000E+00      1.0000000E+00
```

SAMPLE OUTPUT

Page 2

X= 0. FEET TYPICAL TURBOJET AIRCRAFT - SEA LEVEL CONDITIONS - EMT MIXING SMOOTH PAGE 1

X/R	DELTA X FEET
0 0.	1.000000E-01

PT	PRESSURE ATMOS	VELOCITY FEET/SEC	TEMPERTURE DEG. KELVIN	DENSITY GM/CM3	Y/R	FLOW ANG DEGREES	VISCOSITY LB-SEC/FT2	MOLWT	PT
1	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	0.	0.	1.000000E-03	2.801600E+01	1
2	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	1.000000E-01	0.	1.000000E-03	2.801600E+01	2
3	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	2.000000E-01	0.	1.000000E-03	2.801600E+01	3
4	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	3.000000E-01	0.	1.000000E-03	2.801600E+01	4
5	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	4.000000E-01	0.	1.000000E-03	2.801600E+01	5
6	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	5.000000E-01	0.	1.000000E-03	2.801600E+01	6
7	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	6.000000E-01	0.	1.000000E-03	2.801600E+01	7
8	1.000000E+00	1.000000E+03	9.000000E+02	3.793422E-04	7.000000E-01	0.	1.000000E-03	2.801600E+01	8
9	1.000000E+00	1.770000E+03	8.800000E+02	3.879556E-04	8.000000E-01	0.	1.000000E-03	2.801600E+01	9
10	1.000000E+00	1.780000E+03	8.500000E+02	4.016564E-04	9.000000E-01	0.	1.000000E-03	2.801600E+01	10
11	1.000000E+00	8.980000E+02	5.500000E+02	6.267418E-04	1.000000E+00	0.	1.000000E-03	2.801600E+01	11
12	1.000000E+00	7.080000E+01	3.300000E+02	1.034570E-03	1.100000E+00	0.	1.000000E-03	2.801600E+01	12
13	1.000000E+00	4.080000E+01	3.200000E+02	1.066300E-03	1.200000E+00	0.	1.000000E-03	2.801600E+01	13
14	1.000000E+00	2.080000E+01	3.100000E+02	1.101316E-03	1.300000E+00	0.	1.000000E-03	2.801600E+01	14
15	1.000000E+00	1.000000E+01	3.000000E+02	1.138327E-03	1.400000E+00	0.	1.000000E-03	2.801600E+01	15
16	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	1.500000E+00	0.	1.000000E-03	2.801600E+01	16
17	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	1.600000E+00	0.	1.000000E-03	2.801600E+01	17
18	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	1.700000E+00	0.	1.000000E-03	2.801600E+01	18
19	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	1.800000E+00	0.	1.000000E-03	2.801600E+01	19
20	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	1.900000E+00	0.	1.000000E-03	2.801600E+01	20
21	1.000000E+00	1.000000E+00	3.000000E+02	1.138327E-03	2.000000E+00	0.	1.000000E-03	2.801600E+01	21

SAMPLE OUTPUT

Page 3

(Note: Only one species present)

X= 0. FEET

PT	Y/R	N2
1	0.00000	1.00000E+00
2	.10000	1.00000E+00
3	.20000	1.00000E+00
4	.30000	1.00000E+00
5	.40000	1.00000E+00
6	.50000	1.00000E+00
7	.60000	1.00000E+00
8	.70000	1.00000E+00
9	.80000	1.00000E+00
10	.90000	1.00000E+00
11	1.00000	1.00000E+00
12	1.10000	1.00000E+00
13	1.20000	1.00000E+00
14	1.30000	1.00000E+00
15	1.40000	1.00000E+00
16	1.50000	1.00000E+00
17	1.60000	1.00000E+00
18	1.70000	1.00000E+00
19	1.80000	1.00000E+00
20	1.90000	1.00000E+00
21	2.00000	1.00000E+00